The GARCH Model

- ARCH models may require many parameters to adequately describe the volatility process of an asset return.

- To reduce the number of parameters, the Generalized ARCH (GARCH) model was developed.
  - The GARCH model now opens the possibility for an MA process on the volatility.
  - Similar to the reasoning behind a finite MA model adequately representing an infinite AR model.

- \( \alpha_i + \beta_i < 1 \) is necessary for the unconditional variance of \( u_t \) to be finite (similar to unit root discussion).

- A well specified model requires that
  \[
  \alpha_i = 0 \forall i > m \\
  \beta_j = 0 \forall j > s
  \]

- If \( s = 0 \), this is a standard ARCH(m) model.

- \( \alpha_i \) and \( \beta_j \) are referred to as ARCH and GARCH parameters, respectively.

- \( \alpha_0 > 0, \alpha_i \geq 0 \)
  \( \beta_j \geq 0 \)
  \( \sum_{i=1}^{\max(m,s)} (\alpha_i + \beta_i) < 1 \)

- The GARCH(m,s) is given by

\[
\begin{align*}
  u_t &= r_t - \mu_t \\
  u_t &= \sigma_t \nu_t \\
  \sigma_t^2 &= \alpha_0 + \sum_{i=1}^{m} \alpha_i u_{t-i}^2 + \sum_{j=1}^{s} \beta_j \sigma_{t-j}^2
\end{align*}
\]

where \( \nu_t \) is a sequence of iid random variables (white noise) with mean 0 and variance 1.0

- Define \( \eta_t = u_t^2 - \sigma_t^2 \) and substitute into the volatility equation:

\[
\begin{align*}
  u_t^2 &= \alpha_0 + \sum_{i=1}^{\max(m,s)} (\alpha_i + \beta_i) u_{t-i}^2 + \eta_t - \sum_{j=1}^{s} \beta_j u_{t-j}^2 \\
  \sigma_t^2 &= \alpha_0 + \sum_{i=1}^{m} \alpha_i u_{t-i}^2 + \sum_{j=1}^{s} \beta_j \sigma_{t-j}^2
\end{align*}
\]

- \( E(\eta_t) = 0, \text{cov}(\eta_t, \eta_{t-j}) = 0 \) for \( j \geq 1 \)

- This is simply a ARMA model for \( u_t^2 \)

\[
E(u_t^2) = \frac{\alpha_0}{1 - \sum_{i=1}^{\max(m,s)} (\alpha_i + \beta_i)}
\]
Consider the GARCH(1,1) model:

\[ \sigma_t^2 = \alpha_0 + \alpha_1 u_{t-1}^2 + \beta_1 \sigma_{t-1}^2 \]

\[ 0 \leq \alpha_1, \beta_1 \leq 1, \quad (\alpha_1 + \beta_1) < 1 \]

- a large \( u_{t-1}^2 \) or \( \sigma_{t-1}^2 \) gives rise to a large \( \sigma_t^2 \)
- a large \( u_{t-1}^2 \) tends to be followed by another large \( u_t^2 \) (volatility clustering)

Forecasts are obtained like a standard ARMA model:

\[ \sigma_{h+1}^2 = \sigma_h^2 (1) = \alpha_0 + \alpha_1 u_h^2 + \beta_1 \sigma_h^2 \]

for multi-step forecasts, use \( u_t^2 = \sigma_t^2 \)

\[ \sigma_{h+1}^2 = \alpha_0 + (\alpha_1 + \beta_1) \sigma_h^2 + \alpha_1 \sigma_t^2 (\nu_t^2 - 1) \]

let \( t = h + 1 \)

\[ \sigma_{h+2}^2 = \alpha_0 + (\alpha_1 + \beta_1) \sigma_{h+1}^2 + \alpha_1 \sigma_{h+1}^2 (\nu_{h+1}^2 - 1) \]

Since \( E (\nu_{h+1}^2 - 1|F_h) = 0 \)

\[ \sigma_h^2 (2) = \alpha_0 + (\alpha_1 + \beta_1) \sigma_h^2 (1) \]

\[ \sigma_h^2 (\ell) = \alpha_0 + (\alpha_1 + \beta_1) \sigma_h^2 (\ell - 1), \; \ell > 1 \]

Repeated substitution of the forecast equation results in

\[ \sigma_h^2 (\ell) = \frac{\alpha_0}{1 - \alpha_1 - \beta_1} + (\alpha_1 + \beta_1)^{\ell - 1} \sigma_h^2 (1) \]

which implies

\[ \sigma_h^2 (\ell) \to \frac{\alpha_0}{1 - \alpha_1 - \beta_1} \text{ as } \ell \to \infty \]

- Volatility forecasts eventually converge to the unconditional variance of \( u_t \)

---

**Figure 17: Illustrative Example**

Monthly excess returns of S&P 500 Index
We need to model the serial correlations of the asset returns first (the mean equation)

- One could fit an MA(1,3):
  \[ r_t = 0.0062 + u_t + 0.094 u_{t-1} - 0.1409 u_{t-3} \]
  \[ \hat{\sigma} = 0.577 \]

- Or a longer AR framework:
  \[ r_t = 0.0062 + 0.087 r_{t-1} - 0.133 r_{t-3} + 0.081 r_{t-5} - \]
  \[ 0.065 r_{t-6} + 0.060 r_{t-8} + 0.062 r_{t-9} + u_t \]
  \[ (0.006) \quad (0.014) \quad (0.000) \quad (0.024) \]
  \[ \hat{\sigma} = 0.574, \quad \hat{\sigma}^2 = 0.0033 \]

The PACF of \( r_t^2 \) suggests that we would require a very long (9 period) ARCH process

- We could instead model a GARCH(1,1) process

\[ u_t = \sigma_t I_t \]
\[ \sigma_t^2 = \alpha_0 + \alpha_1 u_{t-1}^2 + \beta_1 \sigma_{t-1}^2 \]

Jointly with the AR mean equation:

\[ r_t = 0.0074 - 0.034 r_{t-1} + 0.0108 r_{t-3} - 0.072 r_{t-5} + \]
\[ + 0.058 r_{t-6} + 0.010 r_{t-8} - 0.02 r_{t-9} + u_t \]
\[ (0.000) \quad (0.388) \quad (0.0763) \quad (0.07) \]
\[ (0.1234) \quad (0.768) \quad (0.553) \]
\[ \hat{\sigma}^2 = 0.00008 + 0.1152 u_{t-1}^2 + 0.860 \sigma_{t-1}^2 \]
\[ (0.000) \quad (0.000) \quad (0.000) \]
- Removing (newly) insignificant AR terms:

\[
\begin{align*}
 r_t &= 0.0074 - 0.072r_{t-5} + u_t \\
 \sigma_t^2 &= 0.00008 + 0.1165u_{t-1}^2 + 0.858\sigma_{t-1}^2
\end{align*}
\]

- Using our expression for the unconditional variance

\[
\frac{\alpha_0}{1 - \alpha_1 - \beta_1} = \frac{0.00008}{1 - 0.858 - 0.1165} = 0.0031 \approx \sigma^2
\]

```
proc autoreg data=snp;

   model snp = / nlag=(5) garch=(q=1,p=1);

   output out=out cev=vhat residual=R;

run;
```

![Figure 20: A look at \(\sigma_t\) and \(\nu_t = u_t/\sigma_t\)](image1)

![Figure 21: ACF of \(u_t\) for mean equation](image2)
The Integrated GARCH Model (IGARCH)

- Just like any other AR process - it may contain a unit-root
  - IGARCH models are unit-root GARCH models
- Implications of a unit-root on a GARCH process is just like an ARIMA process
  - the impact of past squared shocks on $u_t$ is permanent

The IGARCH(1,1) model:

\[ u_t = \sigma_t \epsilon_t \]
\[ \sigma_t^2 = \alpha_0 + \beta_1 \sigma_{t-1}^2 + (1 - \beta_1)u_{t-1}^2 \]

- Compared with GARCH(1,1)
  \[ \sigma_t^2 = \alpha_0 + \alpha_1 u_{t-1}^2 + \beta_1 \sigma_{t-1}^2 \]

The IGARCH model imposes the unit-root $\alpha_1 - \beta_1 = 1$ as a model restriction

- Our S&P 500 example doesn’t show much of a change because the unit root somewhat appeared in the simple GARCH(1,1)

\[
\begin{align*}
  r_t &= 0.007368 - 0.069 r_{t-5} + \epsilon_t \\
  \sigma_t^2 &= 0.0000495 + 0.1391 u_{t-1}^2 + 0.8609 \sigma_{t-1}^2
\end{align*}
\]

```
proc autoreg data=snp;
  model snp = / nlag=(5) garch=(q=1,p=1,type=integrated);
  output out=out cev=vhat residual=R;
run;
```
The GARCH-M Model

- It is usual in finance to model the return of a security as being dependent on its volatility
  - Think about risk-premiums

- A GARCH-M (GARCH in the mean) model:
  \[ r_t = \mu + c\sigma_t^2 + u_t \]
  \[ u_t = \sigma_t \epsilon_t \]
  \[ \sigma_t^2 = \alpha_0 + \alpha_1 u_{t-1}^2 + \beta_1 \sigma_{t-1}^2 \]
  where \( \mu \) and \( c \) are parameters.

- Risk can be measured in different forms:
  \[ r_t = \mu + \sigma_t + u_t \]
  \[ r_t = \mu + c \log (\sigma_t^2) + u_t \]

- Our S&P example:

  \( \sigma_t^2 \) in the mean
  \[ r_t = 0.004987 - 0.0741 r_{t-5} + 1.2249 \sigma_t^2 + u_t \]
  \[ \sigma_t^2 = 0.0000845 + 0.1175 u_{t-1}^2 + 0.8553 \sigma_{t-1}^2 \]

  proc autoreg data=snp;
  model snp = / nlag=(5) garch=(p=1,q=1,mean=linear);
  run;

  \( \sigma_t \) in the mean (mean=sqrt)
  \[ r_t = 0.000816 - 0.0743 r_{t-5} + 0.1535 \sigma_t + u_t \]
  \[ \sigma_t^2 = 0.0000827 + 0.1168 u_{t-1}^2 + 0.8568 \sigma_{t-1}^2 \]

  \( \log (\sigma_t^2) \) in the mean (mean=log)
  \[ r_t = 0.0335 - 0.0741 r_{t-5} + 0.0041 \log (\sigma_t^2) + u_t \]
  \[ \sigma_t^2 = 0.0000810 + 0.1164 u_{t-1}^2 + 0.8580 \sigma_{t-1}^2 \]
The EGARCH Model

- A major shortcoming of previous models is that financial volatility tend to have asymmetric effects
  - negative shocks may have a different effect than positive shocks
- The Exponential GARCH model (EGARCH) resolves this shortcoming

To setup the EGARCH model, consider a weighted innovation:

$g(\varepsilon_t) = \theta \varepsilon_t + \gamma \left( |\varepsilon_t| - E(|\varepsilon_t|) \right)$

where $\theta$ and $\gamma$ are (real) constants, and $\varepsilon_t$ and $|\varepsilon_t| - E(|\varepsilon_t|)$ are both zero mean iid sequences (Therefore, $E[g(\varepsilon_t)] = 0$).

- If $\varepsilon_t$ is a Gaussian random variable (normal dist.), then $E(|\varepsilon_t|) = \sqrt{2/\pi}$.
- If $\varepsilon_t$ follows a Student-t dist,

$$E(|\varepsilon_t|) = \frac{2\sqrt{v-2} \Gamma((v+1)/2)}{(v-1) \Gamma(v/2) \sqrt{\pi}}$$

This weighted innovation is asymmetric

$g(\varepsilon_t) = \begin{cases} (\theta + \gamma)\varepsilon_t - \gamma(|\varepsilon_t|) & \text{if } \varepsilon_t \geq 0 \\ (\theta - \gamma)\varepsilon_t - \gamma(|\varepsilon_t|) & \text{if } \varepsilon_t < 0 \end{cases}$

- An EGARCH(1,1) can be written as

$$u_t = \sigma_t \varepsilon_t$$

$$\ln \left( \sigma_t^2 \right) = \alpha_0 + \frac{1 + \beta_1 B + \ldots + \beta_{s-1} B^{s-1}}{1 - \alpha_1 B - \ldots - \alpha_mB^m} g(\varepsilon_{t-1})$$

- The numerator & denominator are polynomials with roots (zeros) outside the unit circle (i.e. >1)
- The model considers $\ln \left( \sigma_t^2 \right)$ to relax all of the nonnegativity constraints of the previous models
- $g(\varepsilon_t)$ enables the model to respond asymmetrically to positive and negative past values of $\varepsilon_t$

- An EGARCH(m,s) model can be written as

$$\ln \left( \sigma_t^2 \right) = \alpha_0 + \frac{1 + \beta_1 B + \ldots + \beta_{s-1} B^{s-1}}{1 - \alpha_1 B - \ldots - \alpha_mB^m} g(\varepsilon_{t-1})$$

where $\alpha_s = (1 - \alpha) \alpha_0 - \gamma \sqrt{2/\pi}$
• Using \( u_t = \sigma_t \epsilon_t \)

\[
\sigma_t^2 = \sigma_{t-1}^2 \exp(\alpha_s \{ \exp((\theta+\gamma) \frac{u_{t-1}}{\sigma_{t-1}}) \} \text{ if } u_{t-1} \geq 0 \\
\exp((\theta-\gamma) \frac{u_{t-1}}{\sigma_{t-1}}) \text{ if } u_{t-1} < 0
\]

- The model is asymmetric if \( \theta \neq 0 \)
- The sign of \( \theta \) determines which shock direction has the bigger impact
  * \( \theta < 0 \) \( \rightarrow \) negative shocks have larger impacts

- Suppose \( \theta = -c \)

\[
\begin{align*}
\epsilon_{t-1} > 0 & \rightarrow \alpha_1 (1 - c) \epsilon_{t-1} \\
\epsilon_{t-1} < 0 & \rightarrow \alpha_1 (1 + c) \epsilon_{t-1}
\end{align*}
\]

• SAS formulates the \( \text{EGARCH}(m,s) \) model in an alternative way

\[
\ln (\sigma_t^2) = \alpha_0 + \sum_{i=1}^{\omega} \alpha_i g(\epsilon_{t-i}) + \sum_{j=1}^{\Omega} \beta_j \ln (\sigma_{t-j}^2)
\]

\[
g(\epsilon_t) = \theta \epsilon_t + \gamma [\epsilon_t - E[\epsilon_t]]
\]

\[
\epsilon_t = u_t / \sigma_t
\]

where \( \gamma = 1 \)

• An \( \text{EGARCH}(1,1) \) can be reduced to

\[
\ln (\sigma_t^2) = \alpha_0 + \alpha_1 (\theta \epsilon_{t-1} + |\epsilon_{t-1}| - \sqrt{2/\pi}) + \beta_1 \ln (\sigma_{t-1}^2)
\]

- \( \theta < 0 \) \( \rightarrow \) negative innovations have large impacts on the log variances than positive innovations

• Our S&P 500 example yet again

\[
\begin{array}{l}
\hat{r_t} = 0.006793 - 0.0926 \hat{r}_{t-5} + u_t \\
(0.0001) \quad (0.0034)
\end{array}
\]

\[
\begin{array}{l}
\ln (\hat{\sigma_t}^2) = -0.150 + 0.2204 \left( -0.2772 \epsilon_{t-1} + |\epsilon_{t-1}| - \sqrt{2/\pi} \right) + 0.9740 \ln (\hat{\sigma}_{t-1}^2) \\
(0.0090) \quad (0.0000) \quad (0.0126) \quad (0.0000)
\end{array}
\]

\[
\text{proc autoreg data=snp;}
\]

\[
\text{model} \text{ snp = / nlag=(5) garch=(p=1,q=1,type=exp);}
\]

\[
\text{run; quit;}
\]
The Threshold GARCH Model

- Another way to uncover leverage effects is to write it as a TGARCH(m,s) model
  \[ \sigma_t^2 = \alpha_0 + \sum_{i=1}^{s} (\alpha_i + \gamma_i N_{t-i}) u_{t-i}^2 + \sum_{j=1}^{m} \beta_j \sigma_{t-j}^2 \]
  where \( N_{t-i} \) is an indicator variable for a negative \( u_{t-i}^2 \)
  \[ N_{t-i} = \begin{cases} 
  1 & \text{if } u_{t-i} < 0 \\
  0 & \text{if } u_{t-i} \geq 0 
  \end{cases} \]
  and \( \alpha_i, \gamma_i, \beta_j \) are all non-negative.
  - A positive \( u_{t-i} \) contributes \( \alpha_i u_{t-i}^2 \) to \( \sigma_t^2 \)
  - A negative \( u_{t-i} \) contributes \( (\alpha_i + \gamma_i) u_{t-i}^2 \) to \( \sigma_t^2 \)

Implementing a TGARCH model in SAS is similar to modeling MA errors:

```sas
data tgarch;
    lu = &var2; lh = &var2;
    arch1_plus = 0.1; arch1_minus = 0.1;
    do i = -500 to &nobs ;
    /* TGARCH */
    if lu > 0 then
        h = (&arch0 + arch1_plus*lu + &garch1*sqrt(lh))**2 ;
    else
        h = (&arch0 + arch1_minus*lu + &garch1*sqrt(lh))**2 ;
    u = sqrt(h) * rannor(1234) ;
    y = &intercept + u;
    lu = u; lh = h;
    if i > 0 then output;
end;
run;/* Estimate Threshold Garch (TGARCH) Model */
```

- The model generally uses zero as the threshold
  - positive vs. negative

- Other thresholds can be entertained:
  \( N_{t-i} = \begin{cases} 
  1 & \text{if } u_{t-i} < c \\
  0 & \text{if } u_{t-i} \geq c 
  \end{cases} \)
  If \( c \) is some negative number, then you would be comparing large negative shocks vs. everything else.
proc model data = tgarch;

parms arch0 .1 arch1_plus .1 arch1_minus .1 garch1 .75;

/* mean model */
y = intercept;

/* variance model */
if zlag(resid.y) < 0 then
   h.y = (arch0 + arch1_plus*zlag(-resid.y) + garch1*zlag(sqrt(h.y)))**2;
else
   h.y = (arch0 + arch1_minus*zlag(-resid.y) + garch1*zlag(sqrt(h.y)))**2;

/* fit the model */
fit y / method = marquardt fiml;
run; quit;

Hope that you won’t have to